An Obstacle Problem for Elasticae

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Euler's Elastic Energy

Let $\lambda \geq 0$. We consider the (length-penalized) elastic bending energy

$$\mathcal{E}_\lambda(\gamma) := \int_\gamma |\kappa_\gamma|^2 \, \mathrm{d}s_\gamma + \lambda \mathcal{L}(\gamma),$$

where $\gamma \in W^{2,2}((0,1);\mathbb{R}^2)$ is an immersed curve with length $\mathcal{L}(\gamma)$, arclength element $\mathrm{d}s_\gamma = |\gamma'(t)| \, \mathrm{d}t$ and curvature vector $\kappa_\gamma := \partial_s^2 \gamma$.

The parameter λ is called *length penalization parameter*. In absence of length penalization (i.e. $\lambda = 0$) the energy $\mathcal{E}_0$ prefers large curves. Indeed, for $\sigma > 0$ one has the *scaling behavior* $\mathcal{E}_0(\sigma \cdot \gamma) = \sigma^{-1}\mathcal{E}_0(\gamma)$. This accounts for a *loss of compactness*, which necessitates length penalization for most variational problems.

Critical points of $\mathcal{E}_\lambda$ are given by pieces of the infamous *elastic curves of Euler* (found in 1744). They satisfy $\partial_s^2 \kappa_\gamma + \frac{1}{2}|\kappa_\gamma|^2 \kappa_\gamma - \lambda \kappa_\gamma = 0$.

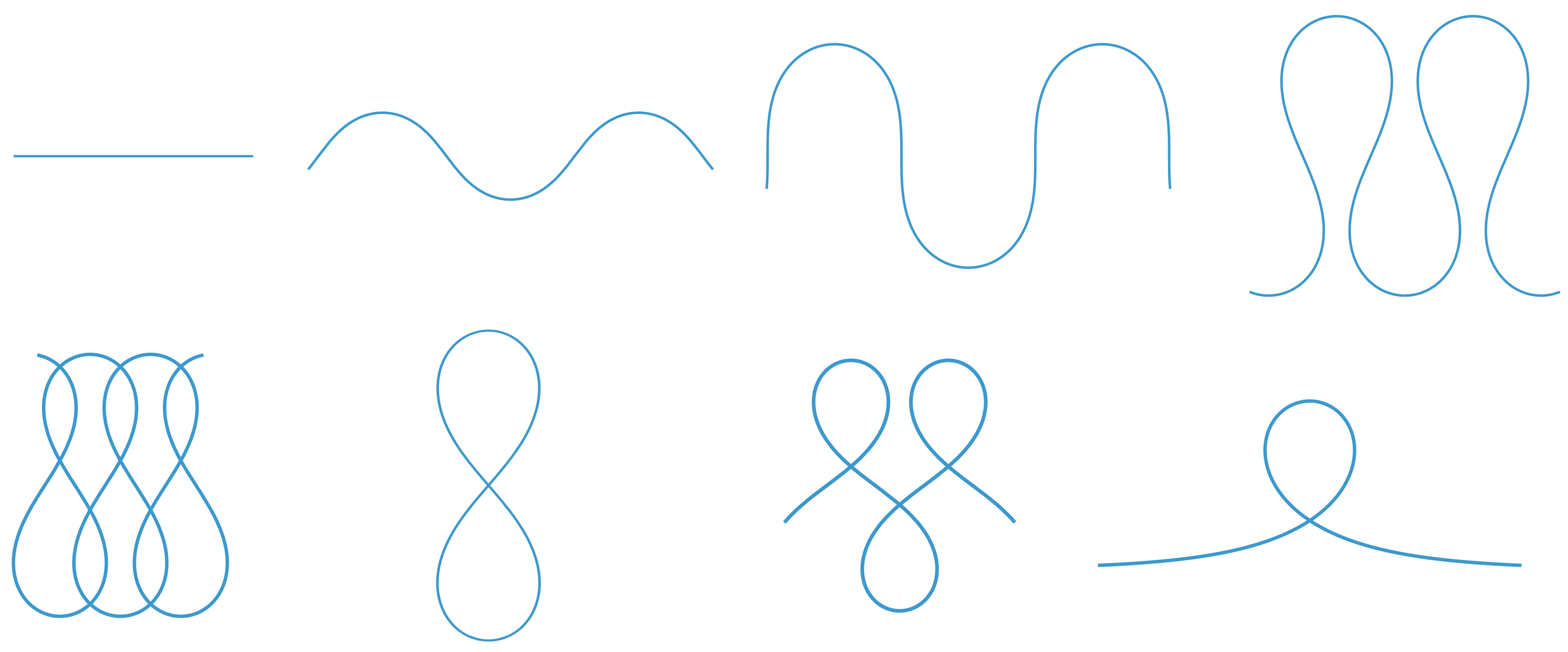

Figure 1: A tour through Euler's elasticae

Our Obstacle Problem

Let $\psi : \mathbb{R} \to \mathbb{R}$ be uniformly continuous s.t. $\psi|_{(-\infty,0)}, \psi|_{(1,\infty)} < 0$, $\psi(\frac{1}{2}) > 0$. Consider the *supergraph* $S_\psi := \{(x,y) \in \mathbb{R}^2 : y \geq \psi(x)\}$.

In our study we seek to minimize $\mathcal{E}_\lambda(\gamma)$ among all immersed curves $\gamma \in W^{2,2}_{\mathrm{imm}}((0,1);\mathbb{R}^2)$ with *fixed ends and obstacle constraint*, i.e.

(i) $\gamma(0) = (0,0)^T$ and $\gamma(1) = (1,0)^T$,
(ii) $\gamma((0,1)) \subset S_\psi$ [that is $\gamma_2(t) \geq \psi(\gamma_1(t))$ for all $t \in (0,1)$].

For $\lambda = 0$, minimizers do not exist (scaling!). For $\lambda > 0$ one can always find a minimizer γ^*, so one can ask further questions, such as

(Q1) How large is the *contact set* $\{t \in (0,1) : \gamma_2(t) = \psi(\gamma_1(t))\}$?
(Q2) What is the *optimal regularity* of γ^*?

Understanding the contact set is vital also for (Q2): At contact points not all perturbations are admissible, which is why Euler-Lagrange methods (and thus regularity) break down on the contact set.

Main Theorem

Our main result deals with the dependence of the contact set on λ.

Theorem. Let $\lambda > 0$ and γ^* minimize our obstacle problem for $\mathcal{E}_\lambda$ with obstacle ψ. Define $C_\psi(\gamma^*) := \{t \in (0,1) : \gamma_2(t) = \psi(\gamma_1(t))\}$.

(a) If $\lambda > \hat{\lambda} \simeq 0.7011$ then γ^* touches the obstacle, i.e. $C_\psi(\gamma^*) \neq \emptyset$. In this case, γ^* is $W^{3,\infty}$-regular (optimal).
(b) If ψ is a *cone obstacle*, there exists $\lambda_0 = \lambda_0(\psi) \in (0,\hat{\lambda})$ such that $\lambda < \lambda_0 \implies C_\psi(\gamma^*) = \emptyset$. Moreover, $\gamma^* \in C^\infty$ is explicit.

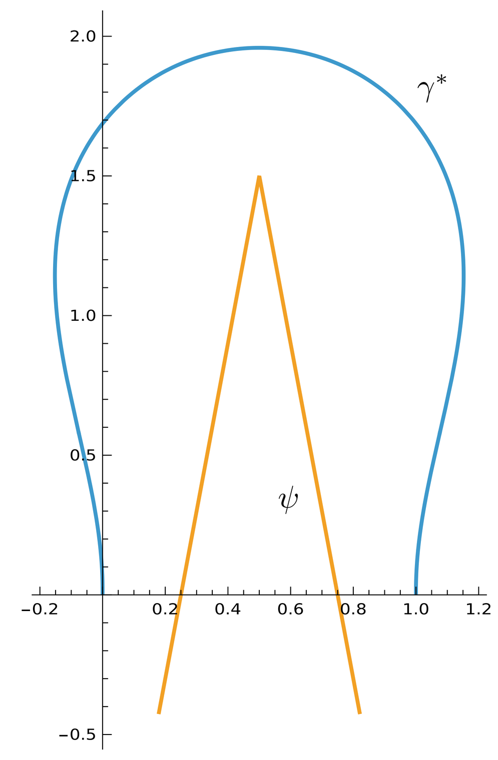

Figure 2: Situation in (b). Which elastica in Figure 1 yields the minimizer?

The fact that the contact set can be empty shows that the problem *lacks a maximum principle* (due to its *higher order* nature).

Proof strategy

Proof of (a). Use *stability analysis*: If $\lambda > \hat{\lambda}$ the only local minimizer of $\mathcal{E}_\lambda$ subject to (i) is the straight line, cf. [2]. But a nontouching minimizer would (by Euler-Lagrange methods) be a local minimizer of $\mathcal{E}_\lambda$!

Proof of (b). We need to rule out touching minimizers. Problem: For small $\lambda > 0$ also touching curves can have very small energy (see Figure 3). Hence, it is not enough to study energy arguments — one has to use properties of minimizers and the *elastica equation* above!

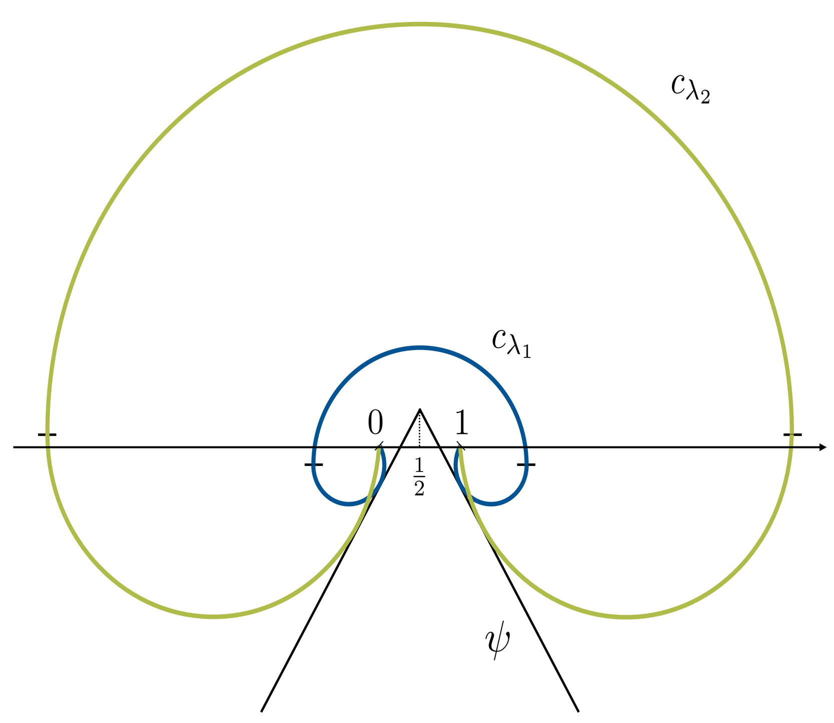

Figure 3: A family of touching curves $(c_\lambda)_{\lambda \in (0,\lambda_0)}$ with $\mathcal{E}_\lambda(c_\lambda) = O(\sqrt{\lambda})$.

[1] M. Müller and K. Yoshizawa. *A nongraphical obstacle problem for elastic curves.* (2025), to appear in Indiana Univ. Math. J.
[2] M. Müller and K. Yoshizawa. *Classification and stability of penalized pinned elasticae.* (2026) J. Differential Equations.